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The Graphic Statics behind the Collier Memorial

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Abstract

This paper reviews the various graphic statics methods applied during the early structural design of the Collier Memorial, Cambridge, MA. Built in 2015 to honor fallen MIT Police Officer Sean Collier, the monument is a vaulted assembly of massive granite blocks. They are held together with pure compression contact when under the action of gravity alone. This non-conventional structural typology has only rare contemporary precedents and its design consequently innovates in various areas. In particular, assumptions of plastic theory and graphic statics have been combined to explore the design space, to assess stability, to study collapse mechanisms and to provide factors of safety related to maximum allowed displacements and maximum allowed live load. These methods proved to be a faster and sounder alternative to conventional discrete-element methods during the conceptual design stage.

Keywords: conceptual structural design, masonry, graphic statics, lower-bound theorem of plastic theory

1. Introduction

The design of daring forms made of self-supporting solid stones is not common practice today. Innovation is required when it comes to tailor methods to assess their stability, especially in today's engineering context where (1) structural design tools must allow for fast, performative and integrated early explorations; (2) superfluous material and redundant technology should be avoided; and (3) strength-and-deflection methods are more widely available than stability methods.

Figure 1: The Collier Memorial (photo credits: Iwan Baan, courtesy of Howëler+Yoon Architects).

This paper focuses on the graphical methods developed during the early form-finding and structural assessment of a 3-meter high, 18-meter wide, vaulted monument made of 32 solid blocks of Virginia Mist granite whose weight ranges from 3 to 10 tons each (Figure 1). The central vault is buttressed by five slender walls (named A, B, C, D, and E) that are anchored in a tension slab and that are approximately 46-centimeters wide and 3.5-meter high. The presented methods allowed the stability of the geometry to be found and assessed at the conceptual stage of the design process. Hence, they permitted subsequent changes in the design to be accommodated without delaying design development (Figure 2). An overview of the design and construction processes is described by Ochsendorf et al. [1].

In this paper, the plastic theory and graphic statics are first briefly reviewed. The following sections then describe the various approaches that the authors developed to assess the stability of the monument: an automated, graphical check to help the architects explore the design space; a detailed verification of static equilibrium using strut-and-tie models; a graphical description of the domains of kinematically admissible wall displacements; a linear formulation of the maximum live load before collapse. Additional methods not presented in this paper included a preliminary check of wall rocking under seismic loads and an explicit definition of the worst directions for seismic shocks. Most of these methods rely on geometrical formulations, and their relevance is discussed in the final section, which also comments on the unavoidable difficulties of interpreting factors of safety when dealing with unconventional masonry structures.

2. Theoretical Background and State of the Art

The precise, actual stress state of a hyperstatic structure is always unknown. Indeed, small displacements and imperfections at the supports, between connections or in the material itself are impossible to predict. In some cases, they may alter the inner state considerably and stresses obtained with conventional elastic theory are irrelevant (Heyman [2]). This statement is even more accurate for highly indeterminate structures like masonry vaults.

This practical issue is avoided when the *lower-bound theorem* of plastic theory is used (Gvozdez [3], translated in Heyman [4]). It states that any load calculated from an equilibrium state that satisfies the yield condition is a lower bound on the value of the collapse load. It can be combined with the *upper-bound theorem* that states that any load calculated from a possible mode of plastic displacement is an upper bound on the value of the collapse load. Both theorems assume a perfect plastic behavior of the material, which in the case of masonry, translates to infinite compressive strength, no tensile strength, and absence of sliding failure (Heyman [5]).

The lower-bound theorem represents an outstanding design tool since it does not require the designer to find the actual stress state that will occur in the structure (Ochsendorf [6]). Moreover, it allows the designer to approximate the stability of a continuous structure through a finite set of struts and ties (Marti [7]). This approach can be related to the thrust lines drawn for the study of masonry arches and domes as early as the 18th Century (Huerta [8]). A thrust line is defined as the locus of the resultants of all the compressive stresses applied on each cross-section of a continuum (Figure 3, top left).

Equilibrium models made of axially loaded rigid bars are here worked by means of graphic statics (Rankine [9], Maxwell [10]). Graphic statics combine a form diagram, representing the actual geometry of the model in static equilibrium, made of point loads and bars in compression and tension, with a force diagram, composed of an equal number of load vectors and bars, the length of which equals the corresponding force magnitude. Recent research in the field has shown that the use of graphic statics can greatly ease the form-finding [11], the modeling [12] [13], and the optimization [14] of structures. In particular, thrust-line analysis [15] and thrust-network analysis [16], allow the interactive form-finding of vaulted structures. A number of original masonry vaults have been designed in recent years thanks to these methods (see for example [17], [18], [19], and [20]). The Collier Memorial differs from those examples in that the blocks are proportionally bigger and fewer in number.

This paper revisits and augments the approach with new thrust line methods. Over the course of the design development of the monument, these methods have been complemented by physical models, thrust network models and distinct element models.

3. Design Space Exploration

An algorithm that generates thrust lines was first developed to allow the fast exploration of stable geometries by the architects (Figure 2). The architects' intention was to build five slender walls buttressing a central vault in stone. The corresponding set of possible geometries is infinite. Yet, only a subset of it will lead to structures whose stability only relies on surface contact and self-weight.

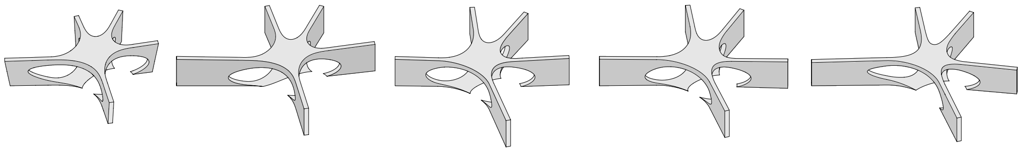

Figure 2: Five solutions among the hundreds of shapes in equilibrium generated with the Grasshopper definition.

Applying the lower-bound theorem of plasticity, a masonry system is stable if there exists a thrust line that (1) is in static equilibrium with the weight of the blocks and that (2) always remains inside the shape of the system. There is generally an infinity of thrust lines for a given masonry shape, i.e. for a given *wall* of the monument. The starting point of the thrust line, its horizontal thrust magnitude, and the geometry of the blocks of the considered wall, are the inputs of the algorithm. The algorithm then computes the weights of the blocks and builds the thrust line segment by segment. Its implementation into a Grasshopper component (within Rhino3D by Robert McNeel and Associates) automated the construction and the display of the thrust lines on top of the wall geometry (Figure 3, top left). The user then could check whether the shape was stable, and gained a sense of how to alter the geometry of an unstable shape in order to make it become statically stable.

Global equilibrium of the vault is guaranteed if thrusts from each wall meet in one point and if they are balanced. This rapid visual check (Figure 3, right) either confirmed stability or led to the modification of the orientation of the walls in the plane.

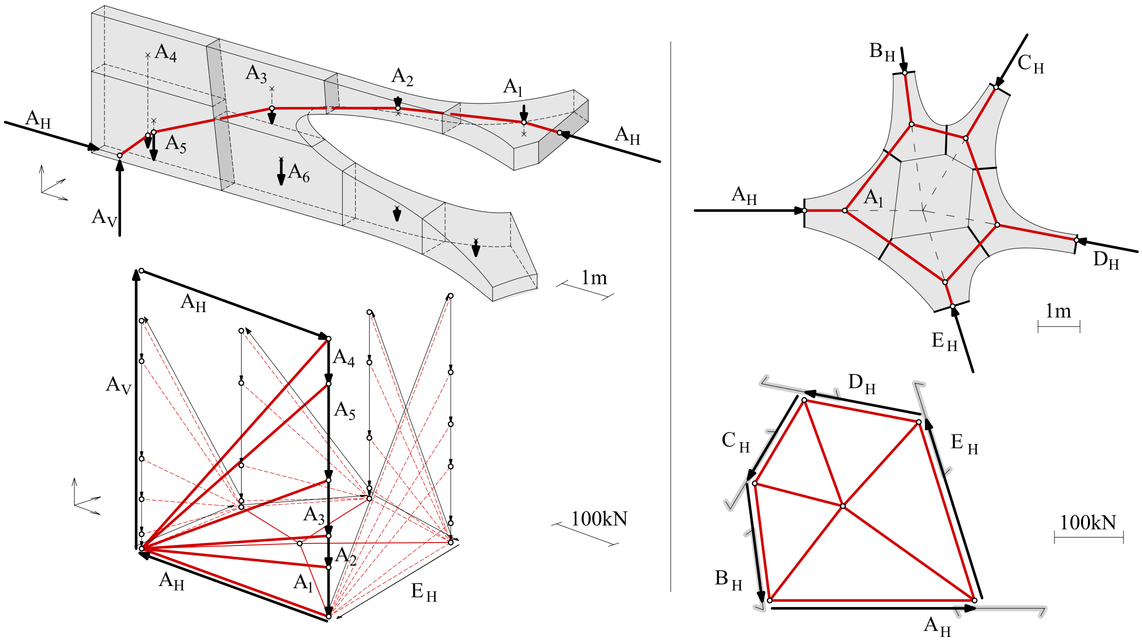

Figure 3: Top left: a feasible thrust line in wall A and (bottom left) its force diagram together with the force polygons of the other walls and the central part. Right: top view of the central blocks and force diagram.

Every set of thrust lines is bound by lines of minimum and maximum thrust. Since the minimum thrust line is the most likely to occur [5], its geometry can be used to define the position and angle of preferential cutting joints (Figure 3, top left). Angles are chosen perpendicular to the thrust line, and critical joints are placed where the thrust line is closest to the edge of the wall — i.e. where hinges are the most likely to happen, in order to let the structure develop hinges naturally without fracturing the brittle stone.

Later structural analyses confirmed that the geometry developed during this elementary structural exploration was safe under other load-cases and dynamic excitations. Real time monitoring of the internal forces and displacements during the decentering process demonstrated the stability of the vault under gravity loads.

4. Refined thrust-line analysis

A refined thrust-line analysis was performed in order to provide factors of safety and to reduce a number of assumptions made during the early exploration. Three additional load cases are considered. They combine self-weight with a 4.8kN/m² (100psf) live load that is distributed on the full vault, distributed on the central vault, or asymmetrically distributed on half of it. Minimum and maximum thrust lines are computed for each wall (Figure 4). This range is further reduced to guarantee global equilibrium at the keystone (grey segments on Figure 3 bottom right).

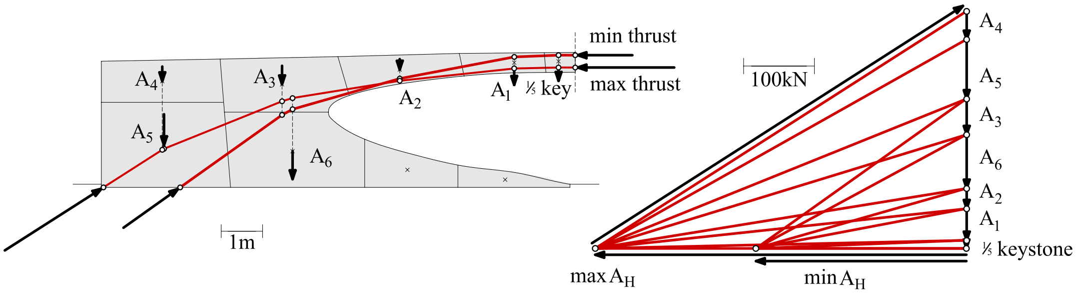

Figure 4: Minimum and maximum thrust lines for wall A (self-weight only).

The weight of the keystone is distributed among the five walls in a way that is unknown due to the static indeterminacy. However, postulations about this distribution are required in order to generate thrust lines that are in equilibrium with the keystone. Two distributions are considered and the least favorable is kept: each wall supports either the full volume of the keystone or a fifth of it if divided by lines joining its centroid to its edges.

Three assumptions made earlier during the interactive form exploration are now checked: (1) a sufficient area of stone must surround the load path so that the stress can be withstood by the material; (2) no tension is transferred between blocks; (3) forces transferred from one block to another should not exceed the friction capacity of the joint (stones should not slide).

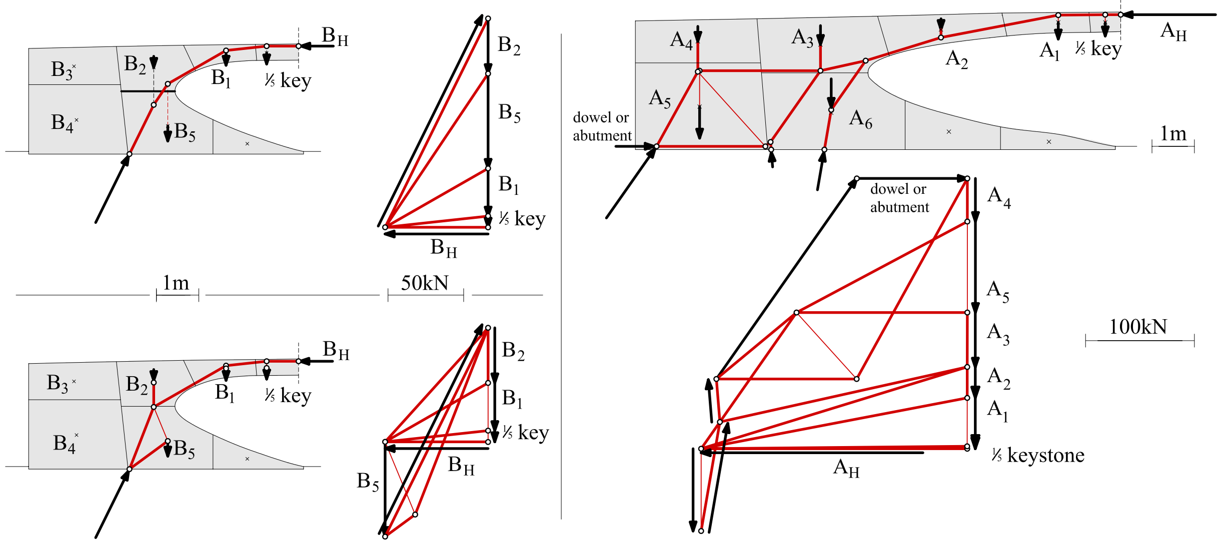

Figure 5: Top left: minimum thrust line in wall B; bottom left: refined strut-and-tie model with same weights and thrust magnitude; right: refined strut-and-tie model in wall A.

The first condition is found to be irrelevant. Given the compressive strength of the granite (290 MPa), the maximum compressive stress (250kN under self-weight), and the thickness of each wall (46cm), the required width of the stress field is always smaller than 2mm, or .8cm if a factor of safety of 4 is applied. This demonstrates that the safety of the monument is governed by stability and not stress.

The second condition is satisfied by constructing alternative strut-and-tie networks where tension ties are not crossing joints between blocks and where the magnitude of these tension ties do not exceed the tensile strength of the stone (Figure 5). This can be assumed due to the high strength of the granite.

The third condition is satisfied by ensuring that the angle of each strut reaching a joint remains within the limit of the friction coefficient related to two faces of granite in contact [21] — i.e. the angle between the strut and the plane of the joint is smaller than 35°. Constructing the form diagram and the force diagram simultaneously eases the definition of these complex strut-and-tie networks.

Results show that the three initial assumptions were safe because “correct” alternative strut-and-tie networks in equilibrium always exist for the same ranges of thrust magnitude.

Moreover, the knowledge acquired with these models, and more precisely the position and orientation of the most-likely ground reactions under self-weight, reduces the complexity of (1) the positioning of the foundation piles supporting the structure and (2) the design and sizing of the connections that will transfer shear forces from the vault to the ground.

5. Collapse Displacements

Assuming that the stones move as rigid bodies, joint opening is mainly due to construction imperfections or differential settlement of the grade slab over time. The hinges allowing the deformation of the masonry are most likely to occur where the minimum thrust lines are closest to the edge [5]. Given the particular cross-section of the vault, two predominant mechanisms that could lead to collapse are identified. In the first case (Figure 6, top), the rotation of the blocks that support the central vault leads to a rotation of the keystone. In the second case (Figure 6, bottom), the central blocks slide along the keystone while the keystone remains supported by the other walls. The first or second mechanism occurs depending on whether an equilibrium state in the keystone can be found when the wall is not supporting it.

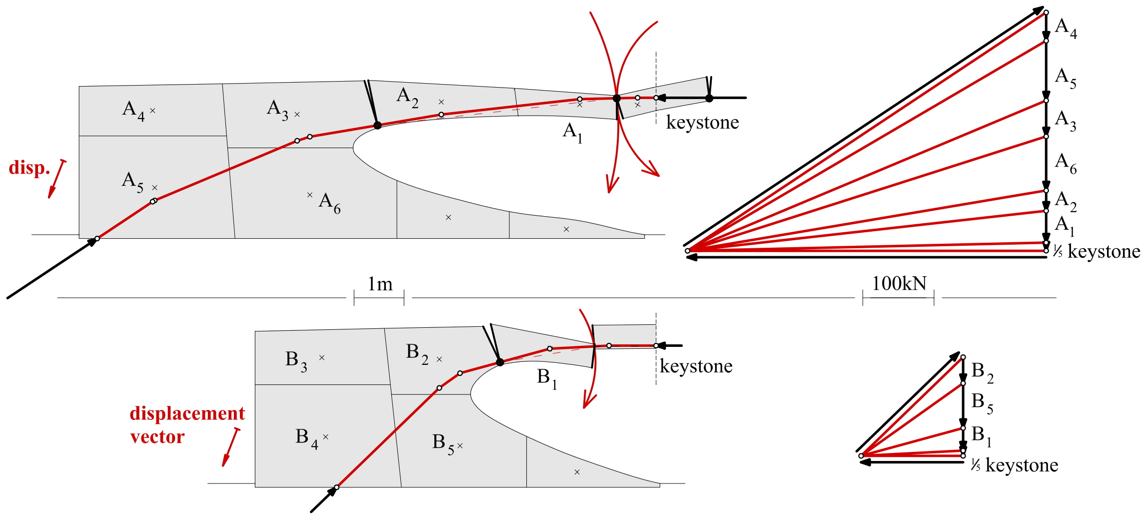

Figure 6: Identified mechanisms of joint opening.
Top: hinging with keystone in wall A; bottom: sliding along the keystone in wall B.

Geometric constructions of the rigid body mechanisms help determine the maximum allowable displacements, a critical concern for a shallow vault. At first glance, they inform that for both types of mechanisms, a horizontal displacement causes a joint opening of the same distance. If this geometric model is parameterized such that the maximum joint opening is expressed as a function of the horizontal and vertical displacements, domains of allowable movements [22] and corresponding maximum joint openings can be obtained (Figure 7). The resulting values become design criteria to limit the predicted elastic deformation of the grade slab.

Superimposing the corresponding thrust lines on the deformed geometry shows that for the given vault geometry (Figure 6), collapse of the structure due to grade slab movement always occurs because of a loss of contact between stones rather than a loss of equilibrium.

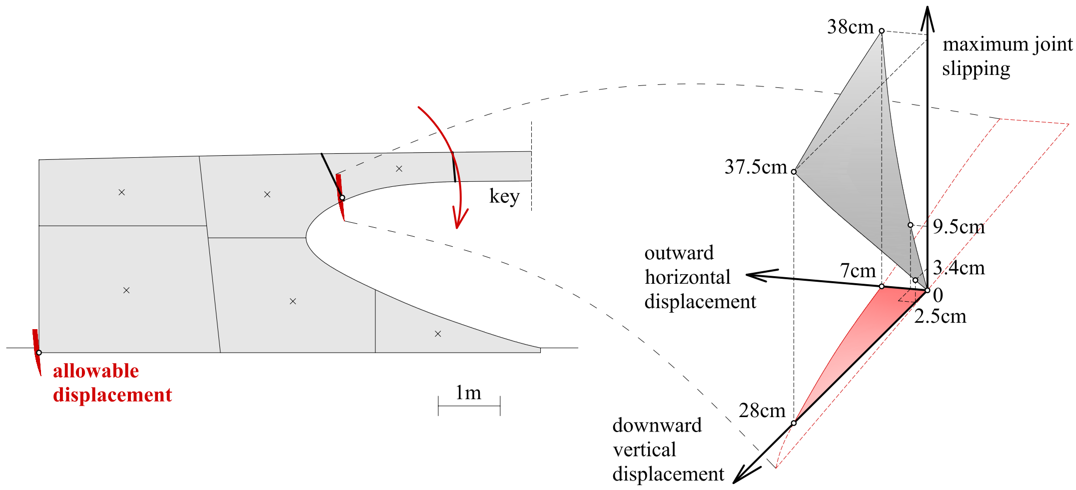

Figure 7: In red, domain of allowable displacements of wall B, demonstrating the high capacity for vertical displacement. In grey on the right, corresponding maximum joint opening between two blocks.

6. Collapse Live Loads

Parametric equilibrium equations applied on assumed collapse mechanisms have been used to determine the maximum live load that is allowed immediately before collapse of the structure, assuming an initial outward displacement of the foundation of ¼” (6.4 mm) vertically and ¼” (6.4mm) horizontally.

As an illustration, Figure 8 shows the domain of possible couples H (horizontal thrust) and LL (live load applied on central blocks of the vault) satisfying the four conditions ensuring the stability of wall A:

- (1) The thrust value must be sufficiently low in order to prevent overturning of the entire wall;
- (2) The thrust value must be sufficiently high in order to prevent collapse of the half-arch;
- (3) The stress applied by the thrust must be lower than the compressive strength of the granite;
- (4) The shear stresses must be lower than the shear strength of the granite, assuming that the shear capacity of the connections with the grade slab are sufficiently high.

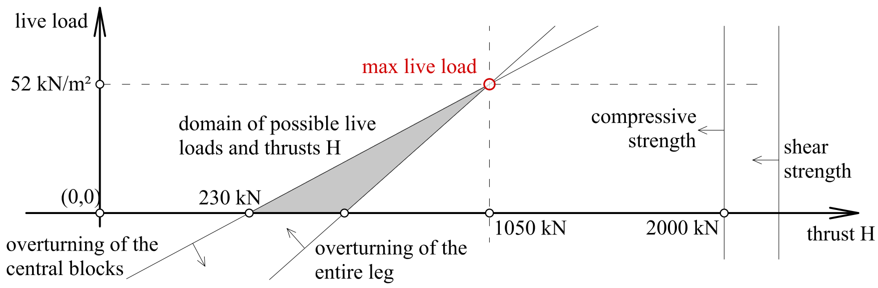

Figure 8: Domain of possible live loads and thrusts before collapse of wall A, after applying outward displacements of ¼” (6.4 mm) in each direction.

It can be seen that values of live load are bounded and that a maximum value exists. Figure 8 also shows that the collapse of wall A is caused by the overturning of the central blocks when the live load is increased. Comparing the results for every wall, a horizontal equilibrium of the entire structure is found when subjected to a live load of 52 kN/m², which is almost 11 times more than the design value of live load (4.8 kN/m²). This high factor of safety is found to be a safe approximation of that obtained by more precise but slower numerical discrete-element based analyses.

7. Discussion

The graphical methods presented in this paper speed up the design and analysis of the presented compression-only structure. By extension, similar methods can speed the design and analysis of other projects governed by structural considerations. Not only do these assumptions and simplified methods ensure the safety of the structure early in the design process, but they also provide a series of factors of safety. The available factors of safety are related to the crushing of stone, collapse live load, collapse displacements, joint openings, and overturning of the walls under seismic excitation (not presented in this paper). Their values range between 10 and 28. When computed by numerical discrete-element software, the same factors of safety range from 20 to 41, meaning that the initial theoretical assumptions are on the conservative side.

However, it has to be recalled that both types of analyses assume a material and a geometry that are perfect, which is naturally not the case in reality. Moreover, analyses are extremely sensitive to imperfections, meaning that high factors of safety are usually less relevant for the designer than safer theoretical assumptions. For that reason, the authors argue for the complementary use of analysis methods that are based on different theoretical assumptions, including physical models. Equilibrium methods based on graphic statics provide a powerful approach for relating form and forces.

8. Conclusion

This paper details new graphical methods and theoretical simplifications to enhance the early structural design of an unconventional compression-based masonry monument made of relatively large blocks.

A distinction is first drawn between conventional elastic assumptions and plastic assumptions. Section 3 describes how automated graphic statics based on highly simplifying assumptions helped the designers quickly explore a large set of stable geometries. The method in section 4 refines the assumptions in order to check stability. The following two sections introduce methods that enable critical factors of safety to be rapidly checked for induced displacements and applied live load magnitudes. A discussion compares the produced factors of safety with those computed by discrete solid-based numerical analyses and argues that both are complementary. The original corpus of methods also included the application of graphic statics to identify critical earthquake directions (not presented in this paper).

On a final note, the geometry of the vault chosen at the very beginning of the design process, i.e. during design space exploration (Section 3), was safe enough to be constructed despite the high number of theoretical assumptions. On the construction site, when the scaffolding supporting the central vault were lowered until the maximum measured differential joint opening reached 6 mm, 96% of the total self-weight of the vault was transferred by pure thrust action (compressive contact only). This successful result is directly related to the nature of the assumptions and methods employed early on.

Thrust-line based methods are commonly employed among engineers in charge of historic building conservation. This paper reveals how similar methods are as convenient for the design of new compression-only structures.

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